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CHARACTERISTICS OF TWO PARTIAL DIFFERENTIAL EQUATIONS OF ORDER ONE*

BY C. A. NOBLE

Given two partial differential equations of order one:

$$F(x, y;\ z, \bar{z}, p, \bar{p}, q, \bar{q}) = 0, \tag{1}$$

$$G(x, y;\ z, \bar{z}, p, \bar{p}, q, \bar{q}) = 0, \tag{2}$$

where x, y are independent variables; z, $\bar{z}$ functions of these variables; and where p, $\bar{p}$, q, $\bar{q}$ denote $\partial z/\partial x$, $\partial \bar{z}/\partial x$, $\partial z/\partial y$, and $\partial \bar{z}/\partial y$, respectively. Assume that F and G are analytic in all the arguments, in the neighborhood of x_0, y_0, z_0, $\bar{z}_0$, p_0, $\bar{p}_0$, q_0, $\bar{q}_0$, where

$$F(x_0, y_0, z_0, \bar{z}_0, p_0, \bar{p}_0, q_0, \bar{q}_0) = 0,$$

$$G(x_0, y_0, z_0, \bar{z}_0, p_0, \bar{p}_0, q_0, \bar{q}_0) = 0;$$

and assume, furthermore, that $F = 0$, $G = 0$ can be written in the normal form

$$q = f(x, y;\ z, \bar{z}, p, \bar{p}),$$

$$\bar{q} = g(x, y;\ z, \bar{z}, p, \bar{p}),$$

where f and g are analytic in all their arguments.

According to the Cauchy-Kowalewski analysis, there will be two, and only two analytic functions:

$$z = \phi(x, y), \tag{3}$$

$$\bar{z} = \bar{\phi}(x, y), \tag{4}$$

which satisfy (1) and (2) simultaneously, and which satisfy also the initial conditions

$$[z]_{y=y_0} = \psi(x), \qquad [\bar{z}]_{y=y_0} = \bar{\psi}(x),$$

where ψ, $\bar{\psi}$ are arbitrary analytic functions.

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Geometrically, this means that a unique pair of surfaces, belonging simultaneously to (1) and (2), can always be passed through the curves

$$y = y_0,\ z = \psi(x) \qquad \text{and} \qquad y = y_0,\ \bar{z} = \bar{\psi}(x).$$

Characteristics, in the present case, according to Hilbert's method of approach, would be such space curves that the Cauchy-Kowalewski analysis would not permit the inference that they determine a unique pair of surfaces of (1) and (2).

If (3), (4) are a surface pair of (1), (2), then, *in general*, the analytic curves determined upon these two surfaces by any arbitrarily selected cylinder

$$y = \lambda(x)$$

will uniquely determine the surfaces (3), (4). We seek those exceptional curves, if any exist, for which the Cauchy-Kowalewski theorem fails. In order to apply this theorem to the given equations, let us make the substitution

$$x = x,\ \eta = y - \lambda(x),$$

$$z = \phi[x,\ \eta + \lambda(x)] = \zeta(x,\ \eta),$$

$$\bar{z} = \bar{\phi}[x,\ \eta + \lambda(x)] = \bar{\zeta}(x,\ \eta).$$

In the new variables, x, η, ζ, $\bar{\zeta}$, the equations (1), (2) become

$$(1^*) \qquad F\{x,\ \eta + \lambda(x),\ \zeta,\ \bar{\zeta},\ \zeta_x - \lambda'\zeta_\eta,\ \bar{\zeta}_x - \lambda'\bar{\zeta}_\eta,\ \zeta_\eta,\ \bar{\zeta}_\eta\} = 0,$$

$$(2^*) \qquad G\{x,\ \eta + \lambda(x),\ \zeta,\ \bar{\zeta},\ \zeta_x - \lambda'\zeta_\eta,\ \bar{\zeta}_x - \lambda'\bar{\zeta}_\eta,\ \zeta_\eta,\ \bar{\zeta}_\eta\} = 0,$$

where the accents denote differentiation with respect to x.

The necessary and sufficient condition that the Cauchy-Kowalewski inference shall be invalid for the curves cut from (3) and (4) by $y = \lambda(x)$, i.e. from $\zeta = \zeta(x,\ \eta)$ and $\bar{\zeta} = \bar{\zeta}(x,\ \eta)$ by $\eta = 0$, is that (1*), (2*) shall not be solvable for ζ_η, $\bar{\zeta}_\eta$; that is, that

$$(5) \qquad \begin{vmatrix} \partial F/\partial\zeta_\eta, & \partial F/\partial\bar{\zeta}_\eta \\ \partial G/\partial\zeta_\eta, & \partial G/\partial\bar{\zeta}_\eta \end{vmatrix} = \begin{vmatrix} F_q - \lambda' F_p, & F_{\bar{q}} - \lambda' F_{\bar{p}} \\ G_q - \lambda' G_p, & G_{\bar{q}}, - \lambda' G_{\bar{p}} \end{vmatrix} = 0,$$

identically in x when for η, y, z, and $\bar{z}$ we write 0, $\lambda(x)$, $\phi[x,\ \lambda(x)]$, and $\bar{\phi}[x,\ \lambda(x)]$ respectively. If, in equation (5), we write $z = \phi(x,\ y)$ and

$\bar{z} = \bar{\phi}(x, y)$, but leave y undisturbed and set $\lambda' = y'$, it becomes an ordinary differential equation in x, y, and determines two one-parameter families of curves lying upon the surface $z = \phi(x, y)$ and two one-parameter families of curves lying upon $\bar{z} = \bar{\phi}(x, y)$. Whether, or not, these curves are actually curves of indetermination as to integral surfaces passing through them, remains still to be answered.

We can set up the system of ordinary differential equations for the independent determination of the characteristics, i.e. without assuming knowledge of the surfaces $z = \phi(x, y)$, $\bar{z} = \bar{\phi}(x, y)$. We have, obviously, in addition to (5),

$$(6) \qquad z' = p + qy', \qquad (7) \qquad \bar{z}' = \bar{p} + \bar{q}y',$$

$$(8) \qquad p' = r + sy', \qquad (9) \qquad \bar{p}' = \bar{r} + \bar{s}y',$$

$$(10) \qquad q' = s + ty', \qquad (11) \qquad \bar{q}' = \bar{s} + \bar{t}y',$$

where the r, s, t have the usual significance.

If we differentiate (1) and (2) with respect to x regarding z, $\bar{z}$, p, $\bar{p}$, q, $\bar{q}$ as functions of x and y, eliminate r, $\bar{r}$, from the two resulting equations by the aid of (8) and (9), multiply the first by $G_q - y'G_p$, the second by $F_q - y'F_p$, and subtract, s and $\bar{s}$ will disappear by virtue of (5), and the single equation results

$$(12) \qquad (F_x + pF_z + p'F_p + \bar{p}F_{\bar{z}} + \bar{p}'F_{\bar{p}})\,(G_q - y'G_p)$$
$$- (G_x + pG_z + p'G_p + \bar{p}G_{\bar{z}} + \bar{p}'G_{\bar{p}})(F_q - y'F_p) = 0.$$

If we differentiate (1), (2) with respect to y, and proceed analogously, we obtain the further equation

$$(13) \qquad (F_y + qF_z + q'F_p + \bar{q}F_{\bar{z}} + \bar{q}'F_{\bar{p}})(G_q - y'G_p)$$
$$- (G_y + qG_z + q'G_p + \bar{q}F_{\bar{z}} + \bar{q}'G_{\bar{p}})(F_q - y'F_p) = 0.$$

Equations (1), (2), (5), (6), (7), (12), (13) suffice, in general, to determine y, z, $\bar{z}$, p, $\bar{p}$, q, $\bar{q}$ as functions of x. With initial values as indicated in the first paragraph, we should have*

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* NOTE. There would be two such sets of equations as (14) and (15), since equation (5) is a quadratic in y'.

$$(14)\qquad \begin{cases} y = y\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \\ z = z\ \}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \\ p = p\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \\ q = q\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \end{cases}$$

$$(15)\qquad \begin{cases} y = y\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \\ \bar{z} = \bar{z}\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \\ \bar{p} = \bar{p}\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{, \\ \bar{q} = \bar{q}\}x;\ x_0,\ y_0,\ z_0,\ \bar{z}_0, p_0, \bar{p}_0,\ q_0,\ \bar{q}_0\{. \end{cases}$$

Equations (14) give the characteristic strip passing through (x_0, y_0, z_0) on the surface $z = \phi(x, y)$, that is, the characteristic curve, together with the tangent plane to the surface at every point of that curve. Equations (15) give the characteristic strip lying upon the surface $\bar{z} = \bar{\phi}(x, y)$ and passing through $(x_0, y_0, \bar{z}_0)$.

Consider now the two integral surfaces (3), (4) passing respectively through the curves

$$y = y_0, \quad z = \psi(x), \quad \text{and} \qquad y = y_0, \quad \bar{z} = \bar{\psi}(x).$$

Assuming that $F = 0$, $G = 0$ can be thrown into the normal form, and that the curves above selected are not characteristics, there will be no other analytic integral surface of (1), (2) through either of these curves. If we select the two elements

$$x_0,\ y_0,\ z_0 = \psi(x_0),\ q_0, p_0 = \psi'(x_0) \text{ and } x_0,\ y_0,\ \bar{z}_0 = \bar{\psi}(x_0),\ \bar{q}_0, \bar{p}_0 = \bar{\psi}'(x_0)$$

we see by (14), (15) that the characteristic strips are completely determined as lying upon their respective surfaces. If we set $x_0 = \xi$ and think of ξ as a continuous variable, these two characteristic strips will, in their motion, generate the surfaces (1), (2) respectively, upon which they lie. But $z = \psi(x)$ was selected as any analytic curve in the plane $y = y_0$ which was not a characteristic, and similarly for $\bar{z} = \bar{\psi}(x)$, i.e., $\psi(\xi)$, $\bar{\psi}(\xi)$ may be thought of as arbitrary analytic functions. The equations (14), (15) then become

$$(16)\qquad \begin{cases} y = y\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \\ z = z\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \\ p = p\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \\ q = q\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \end{cases}$$

$$(17)\qquad \begin{cases} y = y\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \\ \bar{z} = \bar{z}\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \\ \bar{p} = p\{x;\ \xi,\ y_0,\ \psi(\xi),\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}, \\ \bar{q} = \bar{q}\{x;\ \xi,\ y_0,\ \psi(\xi)\ \bar{\psi}(\xi),\ \psi'(\xi),\ \bar{\psi}'(\xi),\ q_0,\ \bar{q}_0\}. \end{cases}$$

The first two equations in (16) and in (17) may be looked upon as furnishing the general solution of (1) and (2), expressed in terms of the parameter ξ. The solution involves two arbitrary functions, as one should expect.

The entire foregoing process is an extension of that presented by Hilbert in his lectures in 1900–1901 (see also Hedrick, *Annals of Mathematics*, July, 1903). Here, as there, it appears that the equations of the characteristics suffice to solve completely the proposed partial differential equations. In other words, two partial differential equations of the first order with two unknown functions may be made to depend completely upon a system of ordinary differential equations.

In order to exhibit the characteristics of (1), (2) in relation to the calculus of variations, let us consider the problem of rendering the integral

$$\int_{u_0}^{u_1} \{\lambda(u)(z' - px' - qy') + \mu(u)(\bar{z}' - \bar{p}x' - \bar{q}y')\}\,du$$

an extremum, with the auxiliary conditions

$$F = 0, \qquad G = 0,$$

whereby λ, μ are arbitrary functions, and accents denote differentiation with respect to u. The necessary conditions for the solution of the above problem, the so-called Lagrange equations, proceed from the vanishing of the first variation of the integral

$$\int_{u_0}^{u_1} \{\lambda(u)(z' - px' - qy') + \mu(u)(\bar{z}' - \bar{p}x' - \bar{q}y') + \xi F + \eta G\}\,du,$$

where ξ, η are undetermined functions of u. These equations of condition, twelve in number, are

$$\begin{array}{llll}
(1) & d(\lambda p+\mu\bar{p})/du+\xi F_x+\eta G_x=0 & (2) & d(\lambda q+\mu\bar{q})/du+\xi F_y+\eta G_y=0 \\
(3) & \lambda'-\xi F_z-\eta G_z=0 & (4) & \mu'-\xi F_{\bar{z}}-\eta G_{\bar{z}}=0 \\
(5) & \lambda x'-\xi F_p-\eta G_p=0 & (6) & \mu x'-\xi F_{\bar{p}}-\eta G_{\bar{p}}=0 \\
(7) & \lambda y'-\xi F_q-\eta G_q=0 & (8) & \mu y'-\xi F_{\bar{q}}-\eta G_{\bar{q}}=0 \\
(9) & z'-px'-qy'=0 & (10) & \bar{z}'-\bar{p}x'-\bar{q}y'=0 \\
(11) & F=0 & (12) & G=0
\end{array}$$

By means of (3), (4), (5), (6) we can eliminate λ', μ', λ, μ from (1), (2), (7), (8). The resulting equations are

$$(13)\qquad (y'/x')^2(F_pG_{\bar{p}}-F_{\bar{p}}G_p)-y'/x'(F_pG_{\bar{q}}-F_{\bar{p}}G_q+F_qG_{\bar{p}}-F_{\bar{q}}G_p) \\ +F_qG_{\bar{q}}-F_{\bar{q}}G_q=0$$

$$(14),(15)\qquad \frac{F_x+pF_x+p'/x'F_p+\bar{p}F_{\bar{z}}+\bar{p}'/x'F_{\bar{p}}}{G_x+pG_z+p'/x'G_p+\bar{p}G_{\bar{z}}+\bar{p}'/x'G_{\bar{p}}} \\ =\frac{F_y+qF_z+q'/x'F_p+\bar{q}F_{\bar{z}}+\bar{q}'/x'F_{\bar{p}}}{G_y+qG_z+q'/x'G_p+\bar{q}G_{\bar{z}}+\bar{q}'/x'G_{\bar{p}}}=\frac{y'/x'F_p-F_q}{y'/x'G_p-G_q}$$

The seven equations (9), (10), $\cdots$ (14), (15) suffice to determine the seven ratios $x':y':z':\bar{z}':p':\bar{p}':q':\bar{q}'$; that is, to determine y, z, $\bar{z}$, p, $\bar{p}$, q, $\bar{q}$ as functions of x. They are identical with the seven equations obtained by the first method for determining the characteristics.

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